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Rational Approximation on the Nonnegative Integers

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Recently we [1] have proved

THEOREM 1. *Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_0>0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function. We denote by N the set of all nonnegative integers. Then, for every integer $n \geqslant 2$, there exists a polynomial $P(x)$ of degree at most $2n$ such that*

$$\left\| \frac{1}{f(x)} - \frac{1}{P(x)} \right\|_{L_\infty(N)} \leqslant \frac{2}{f(2n)}. \tag{1}$$

We now prove

THEOREM 2. *Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_0>0$, $a_k \geqslant 0$ $(k \geqslant 1)$, be an entire function of order ρ $(0<\rho<\infty)$, type τ, and lower type ω $(0<\omega \leqslant \tau<\infty)$. Let $0<\varepsilon<1$. There exists an integer $k>1$ such that, for all integers some $n \geqslant n_0(\varepsilon) \geqslant 0$,* if *$P(x)$, $Q(x)$ are real polynomials of degree $\leqslant n$ with nonnegative coefficients $(Q(0)>0)$, then*

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty(N)} \geqslant 4^{-1} f(n^{1/\rho})^{(-\tau(1+\varepsilon))/\omega(1-\varepsilon)} (2k)^{-n}. \tag{2}$$

THEOREM 3. *Let $f(z)=\sum_{k=0}^{\infty} a_k z^k$, $a_0>0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0<\rho<\infty)$, type τ, and lower type ω $(0<\omega \leqslant \tau<\infty)$. Given ε, $0<\varepsilon<1$, there exists a positive integer k such that if n is an integer $\geqslant$ some $n_0(\varepsilon) \geqslant 1$ and if $P(x)$, $Q(x)$ are real polynomials of degree $\leqslant 2n$ $(Q(0) \neq 0)$, then*

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty(N)} \geqslant \frac{n^{-2}(16)^{-2n}(4k+2)^{-4n}}{4[f(n)]^{(\tau(1+\varepsilon))/(\omega(1-\varepsilon))}}. \tag{3}$$

We need the following lemmas.

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LEMMA 1. *Let $P(x)$ be a polynomial of degree at most $2n$ $(n \geqslant 1)$ satisfying the assumption that $|P(k)|$ is bounded by 1, for $k = a, a+1, a+2, \ldots, a+n, \ldots, a+2n$, a being a nonnegative real: Then*

$$\max_{[a,\, a+2n]} |P(x)| \leqslant n(16)^n. \tag{4}$$

This lemma is known for $a = 0$ [1, Lemma 3]. The proof for $a > 0$ is very similar and omitted.

LEMMA 2 [3, p. 68]. *Let $P(x)$ be a real polynomial of degree at most n $(\geqslant 0)$. $|P(x)|$ is bounded by M on an interval $[a, b] \subset [c, d]$, then, throughout $[c, d]$ we have*

$$|P(x)| \leqslant M \left| T_n\left(\frac{2(d-c)}{(b-a)} - 1\right)\right|, \tag{5}$$

where

$$2T_n(x) = (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n.$$

Proof of Theorem 2. For $r \geqslant 0$, $M(r) = \mathrm{Max}_{|Z| = r} |f(z)|$. Then, by assumption,

$$\lim_{r \to \infty}{}^{\sup}_{\inf} \frac{\log M(r)}{r^\rho} = \frac{\tau}{\omega}. \tag{6}$$

Let $n \geqslant 1$ be an integer and let $P(x)$, $Q(x)$ be real polynomials of degree $\leqslant n$ with nonnegative coefficients, $Q(0) > 0$. Set

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty(N)} = \delta. \tag{7}$$

For n large, we choose a positive integer m such that

$$m^\rho \leqslant n < 2m^\rho. \tag{8}$$

Further, we can find an integer $k > 1$ such that

$$1 + 2\log(2k) + \tau(1+\varepsilon) < \omega(1-\varepsilon)\, k^\rho 2^\rho. \tag{9}$$

We require n to be so large that, from (6) and (7),

$$\frac{P(m)}{Q(m)} \geqslant \frac{1}{f(m)} - \delta \geqslant \exp(-m^\rho \tau(1+\varepsilon)) - \delta. \tag{10}$$

On the other hand, we get from (6) and (7), observing the fact that $P(x)$ and $Q(x)$ have only nonnegative real coefficients,

$$\frac{P(m)}{(2k)^n Q(m)} \leqslant \frac{P(2mk)}{Q(2mk)} \leqslant \frac{1}{f(2mk)} + \delta \leqslant \exp(-2^\rho m^\rho k^\rho \omega(1-\varepsilon)) + \delta. \quad (11)$$

From (8)–(11) we get, after some work, for all large n,

$$4\delta(2k)^n \geqslant \exp(-n\tau(1+\varepsilon)). \quad (12)$$

From (12) we get, for all large n,

$$\delta \geqslant (2k)^{-n} 4^{-1} \exp(-n\tau(1+\varepsilon)) \geqslant 4^{-1} [f(n^{1/\rho})]^{(-\tau(1+\varepsilon))/(\omega(1-\varepsilon))} (2k)^{-n},$$

establishing (2).

Remark. Let $f(z)$ satisfy the assumptions of the first sentence of Theorem 2. Then it has been established in [2] that

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^n a_k x^k} \right\|_{L_\infty[0,\infty)}^{1/n} \leqslant \exp\left(\frac{-\omega}{e\rho\tau + \rho\omega}\right).$$

Proof of Theorem 3. Choose a positive integer k satisfying, for $n = 0, 1, 2, \ldots$,

$$2n^2(16)^{2n}(4k+2)^{4n} < \exp\{[(k+1)\,n]^\rho\, \omega(1-\varepsilon) - (2n)^\rho \tau(1+\varepsilon)\}. \quad (13)$$

Let $P(x)$, $Q(x)$ ($Q(0) \neq 0$) be real polynomials of degree $\leqslant 2n$, $n \geqslant 1$, and set

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty(N)} = \delta. \quad (14)$$

Now we normalize $Q(x)$ so that

$$\max_{x \in \{0,1,2,\ldots,2n\}} |Q(x)| = 1. \quad (15)$$

By Lemma 1 with $a = 0$, we get

$$\max_{[0,2n]} |Q(x)| \leqslant n(16)^n. \quad (16)$$

By applying Lemma 2 to (16) over the interval $[0, 2(k+1)n]$, we get

$$\max_{[0,2(k+1)n]} |Q(x)| \leqslant n(16)^n (4k+2)^{2n}. \quad (17)$$

From (15) it is obvious that there is an integer j $(0 \leqslant j \leqslant 2n)$ for which

$$|Q(j)| = 1. \tag{18}$$

By (14),

$$|P(j)| \geqslant \frac{1}{f(j)} - \delta. \tag{19}$$

We can find an $n_0 = n_0(\varepsilon)$ so large such that for all $n \geqslant n_0(\varepsilon)$ we have on $[0, 2n]$ from (19),

$$|P(j)| \geqslant \exp(-(2n)^\rho \tau(1+\varepsilon)) - \delta. \tag{20}$$

On the other hand, we get from (14) on $[(k+1)n, 2(k+1)n]$,

$$\begin{aligned} |P(x)| &\leqslant \max_{[0, 2(k+1)n]} |Q(x)| \left(\frac{1}{f(x)} + \delta\right) \leqslant n(16)^n (4k+2)^{2n} \left(\frac{1}{f(x)} + \delta\right) \\ &\leqslant n(16)^n (4k+2)^{2n} (\exp\{-[(k+1)n)^\rho \omega(1-\varepsilon)\} + \delta). \end{aligned} \tag{21}$$

Equation (21) is valid throughout $[(k+1)n, 2(k+1)n]$ and hence on $[2kn, 2(k+1)n]$. By applying Lemma 1 to (21) over $[2kn, 2(k+1)n]$, we get

$$\max_{[2kn, 2(k+1)n]} |P(x)| \leqslant n^2 (16)^{2n} (4k+2)^{2n} \{\exp(-[(k+1)n]^\rho \omega(1-\varepsilon)) + \delta\}. \tag{22}$$

By applying Lemma 2 to (22) over the interval $[0, 2(k+1)n]$, we get

$$\max_{[0, 2(k+1)n]} |P(x)| \leqslant n^2 (16)^{2n} (4k+2)^{4n} \{\exp(-[(k+1)n]^\rho \omega(1-\varepsilon)) + \delta\}. \tag{23}$$

From (20) and (23) we have

$$\begin{aligned} &\exp(-(2n)^\rho \tau(1+\varepsilon)) - \delta \\ &\qquad \leqslant n^2 (16)^{2n} (4k+2)^{4n} [\exp(-[(k+1)n]^\rho \omega(1-\varepsilon)) + \delta]. \end{aligned} \tag{24}$$

From (13) and (24), we get for all large n,

$$\delta \geqslant \frac{\exp(-(2n)^\rho \tau(1+\varepsilon))}{4n^2 (16)^{2n} (4k+2)^{4n}} \geqslant \frac{n^{-2} (16)^{-2n} (4k+2)^{-4n}}{4[f(2n)]^{(\tau(1+\varepsilon))/(\omega(1-\varepsilon))}},$$

proving the desired inequality.

REFERENCES

1. P. ERDÖS, D. J. NEWMAN, AND A. R. REDDY, Approximations by rational functions, *J. London Math. Soc.* (*2*) **15** (1977), 319–328.
2. A. R. REDDY, A note on rational approximation, *Bull. London Math. Soc.* **8** (1976), 41–44.
3. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," Macmillan Co., New York, 1963.